Math 10A with Professor Stankova
Worksheet, Discussion #40; Monday, 12/4/2017
GSI name: Roy Zhao

Problems

1. Let $f(x) = \sqrt{\sin(x)}$. Find $f'(x)$.

Solution:
$$f'(x) = \frac{\cos(x)}{2\sqrt{\sin(x)}}.$$

2. Find the equation for the tangent line to $(x^2 + y^2 - 2x)^2 = x^2 + y^2$ at the point $(0, 1)$.

Solution: Take the derivative to get an equation for y' in terms of x and y. Then use the line equation to get
$$y = 2x + 1.$$

3. Find $\lim_{x\to\infty} \frac{x+e^{-x}}{2x+1}$.

Solution: Use LHopital's to get the limit is $\frac{1}{2}$.

4. Find $\lim_{x\to\infty} \frac{\sqrt{x}}{\ln x}$.

Solution: Use LHopital's to get ∞.

5. Find $\lim_{x\to 0} \frac{1-\cos(x)}{\sec(x)}$.

Solution: We can just plug in $x = 0$ to get 0.

6. Find $\lim_{x\to 0^+} x^{\sin x}$.

Solution: Write $x^{\sin x} = e^{\ln x \sin(x)}$ and use Lhopital's in the exponent to get $e^0 = 1$.

7. Graph $y = x^4 - 2x^2$.

Solution: Find the limits and critical points. It looks like a W.

8. You are standing at the shore and there is a ball 15 meters along the shore and 6 meters out. You run at a rate of 6.4 m/s and swim at rate of $0.91 m/s$. In order to get to the ball, you run some distance along the shore and then swim in a diagonal path to the ball. How long should you run on the shore to get to the ball the fastest?

Solution: If x is the distance you run, you want to maximize

$$\frac{x}{6.4} + \frac{\sqrt{36 + (15 - x)^2}}{0.91}$$

Doing so gives you $x \approx 14.14$.

9. A cone shaped funnel is 25 cm in height and has a radius 10 at the top. Water is flowing out the bottom at a rate of 2.5 cm^3/sec. What is the rate at which the height h of the water is dropping when $h = 15$.

Solution: Letting r be the radius of the cone at time t, we get that $r/h = R/H = \frac{2}{5}$ by similar triangles. Then since $V = \pi/3r^2h$, we get a formula $V' = \frac{4\pi}{25}h^2h'$. Therefore, at $V' = -2.5$ and $h = 15$ gives us $h' = \frac{-2.5}{36\pi}$.

10. Find the roots of $g(x) = x^3 + x - 5$ (explain how many and how you would find them).

Solution: First we note that $g'(x) = 3x^2 + 1$ so g is always increasing and only can have 1 root. So we would guess a value at $x = 1$ and repeatedly apply

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}.$$

True/False

11. **TRUE** False The vertical line test tests whether a curve in the plane is the graph of a function.

Solution: The vertical line test makes sure there is at most one output to each input.

12. **TRUE** False Integration and differentiation are inverse processes linked by the Fundamental Theorem of Calculus.

Solution: Derivative of an integral is the function itself.

13. True **FALSE** Every one-to-one function has an inverse.

Solution: Functions must be one-to-one and onto(surjective) in order to have an inverse.

14. **TRUE** False Every exponential function has a doubling time.

15. True **FALSE** The doubling time of $y = 3^x$ is a period of this function.

Solution: This function is not periodic and so it does not have a period.

16. **TRUE** False The log-log plot turns a power function into a linear function, but we need a semi-log plot to turn an exponential function into a linear one.

Solution: If $y = ax^b$, then taking logs gives $\log y = b \log x + \log a$, which is linear in a log-log plot. If $y = Ce^{bx}$, then $\log y = bx + \log C$, which is linear in a semi-log plot.

17. True **FALSE** Limit laws are at the bottom of the Calculus structure.

Solution: Limits are at the bottom whereas limit laws are just tools that let us calculate them.

18. **TRUE** False We can always plug in $x = c$ to find the limit $\lim_{x \to c} f(x)$ except when the function is not continuous at $x = c$.

Solution: We can only plug in $x = c$ when the function is continuous there.

19. True **FALSE** If Limit Laws do not apply from the get-go when attempting to find $\lim_{x \to c} f(x)$, then the limit does NOT exist.

Solution: We may have to use PSTs like L'Hopital's or simplifying.

20. **TRUE** False Problem-solving techniques (PSTs) are often used to find limits $\lim_{x \to c} f(x)$ when we have a hard time plugging in $x = c$.

Solution: PSTs can convert an indeterminate limit to a determinate one.

21. **TRUE** False The definition of limit for functions is responsible for the proof of the Limit Laws.
22. True **FALSE** Horizontal asymptotes can NEVER have any points in common with the graph of a function.

Solution: Horizontal asymptotes may have infinitely many points in common with graph of the function (like $\sin(x)/x$).

23. **TRUE** False Vertical asymptotes CAN have common points with the graph of a function.

Solution: A function can have at most one common point with a vertical asymptote.

24. **TRUE** False While a limit $\lim_{x \to c} f(x)$ does not care what happens exactly at $x = c$ because the limit is concerned only with the behavior of $f(x)$ nearby $x = c$, continuity does care about both and wants them to coincide.

Solution: Limits only care about what happens near $x = c$ and continuity means that the two are equal.

25. True **FALSE** There are at least two pieces of the Calculus structure that are responsible for proving DL+, one of which is the definition of continuity.

Solution: To prove DL+, we need to use the definition of a derivative and LL+.

26. **TRUE** False A composition of two continuous functions, as long as it is well-defined, is always continuous.

27. **TRUE** False The tangent slope of a function is the limit of infinitely many secant slopes.

Solution: The tangent is defined as the limit as the secant becomes closer and closer.

28. True **FALSE** In the definition of the derivative $f'(x_0), x_0$ varies while $h \to 0$.

Solution: x_0 does not vary.

29. **TRUE** False When calculating the derivative of a function by the derivative definition, we can never first plug in $h = 0$ because we will inevitably get $\frac{0}{0}$; instead, we must first simplify until we cancel h from top and bottom of the fraction, and only then we can apply LLs or basic limits of well-known functions.

Solution: Plugging in $h = 0$ will give you $0/0$.

30. True **FALSE** There is a geometric and an algebraic interpretations of the derivative, and the two do not always address the same concept for some weird functions.

Solution: They are the same concept (slope at a point).

31. True **FALSE** To prove that $(e^x)' = e^x$, we can reduce the problem to calculating the limit $\lim_{h\to 0} \frac{e^h-1}{h}$, which in turn is precisely the definition of the derivative $(e^x)'|_{x=1}$.

> **Solution:** It is the definition of the derivative $(e^x)'|_{x=0}$.

32. **TRUE** False After we define what derivatives are, we try to get away from the definition of derivative as fast as possible by discovering and proving DLs because it is cumbersome to use the definition all the time; yet, for some complicated functions that do not comform to any of our DLs we are bound to go down again to the basics, start from scratch, and use the definition of derivative.

> **Solution:** Calculating a limit all the time takes a lot of work whereas if we know the derivative of basic functions, we can use the product rule and chain rule to easily calculate the derivative.

33. **TRUE** False Defining e as the only real number a for which $\lim_{h\to 0} \frac{a^h-1}{h} = 1$ is vastly inconvenient for calculating the value of e, while definition e as the limit $e = \lim_{n\to\infty}(1 + 1/n)^n$ is quite practical as it gives us the opportunity to calculate e with whatever precision we like; yet, the two choices are equivalent and yield the same value of e.

34. True **FALSE** Reciprocal functions have inverse derivatives.

> **Solution:** The derivative of a reciprocal function is $-f'(x)/f(x)^2$. But, inverse functions have reciprocal derivatives.

35. True **FALSE** If a function is not differentiable at $x = c$, then it cannot be continuous there either.

> **Solution:** A function can be continuous but not differentiable.

36. **TRUE** False The Chain Rule can be used to provide shortcut formulas for derivatives of reciprocals of functions and, more generally, for derivatives of powers of functions.

Solution: We can use the chain rule to calculate $(f(x)^n)' = n(f(x))^{n-1}f'(x)$.

37. True **FALSE** Implicit differentiation is typically used when we are trying to compute the derivative of a function given by a complex formula $y = f(x)$.

Solution: Usually we use it to calculate the derivative of $f(x, y) = 0$.

38. **TRUE** False Turning $y \mapsto y(x)$ is a suggested (but not absolutely necessary) step when doing implicit differentiation because otherwise it is hard to apply CR and in the process we may incorrectly omit some of the derivatives $y'(x)$.

Solution: It is recommended in order for you to remember we are taking the derivative with respect to x so $\frac{dy}{dx} \neq 1$.

39. True **FALSE** The second derivative test for concavity is NOT a bullet-proof test because in none of the possible 4 cases can we make any definitive conclusions about the function.

Solution: We can make a definitive conclusion (local minimum/maximum) when $f''(x) \neq 0$. But we cannot say anything if $f''(x) = 0$.

40. **TRUE** False If the first derivative changes its sign, we are absolutely sure that the original function has a local extremum at x_0 too.

Solution: If the derivative changes sign, then there is a local extremum.

41. **TRUE** False Using the graph of $f'(x)$, we can sketch many graphs of the possible original functions $f(x)$.

Solution: By integration, there are many antiderivatives and they are differ by a constant (vertical shift).

42. **TRUE** False When x_0 is not in the domain of $f(x)$, we cannot automatically assume that $f(x)$ has a vertical asymptote there; instead, we need to find out what $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$ are and those could be different or non-existent.

43. True **FALSE** As done in class, when constructing the table to study the graph of $f(x)$, we should always include the zeros of $f(x)$ as special inputs as they will never unnecessarily clutter this table, unless such an x_0 is also a zero of the first and/or second derivative of $f(x)$, in which case we should never include such an x_0 in the table.

Solution: You should not include the zeros of $f(x)$ and only the zeros of $f'(x)$, $f''(x)$.

44. True **FALSE** The logarithmic derivative of $f(x)$ explains why the log-log plot of $f(x)$ may be a better representation of the function's rate of change relative to its size.

Solution: A semilog plot shows this better.

45. **TRUE** False The logarithmic derivative of $f(x)$ can also be used as a sleek approach to calculating the derivative of $f(x)$.

Solution: We just need to multiply the logarithm derivative by the function itself again to find the derivative of the original function.

46. **TRUE** False Working with related rates of change is similar to implicit differentiation in that we are finding the derivative of some function without explicitly finding a formula for this function.

47. True **FALSE** When a differentiable function on a closed interval has only one critical point, then the function is bound to have a global maximum or minimum there.

Solution: It could occur at endpoints as well.

48. **TRUE** False When a rational function $f(x)$ has an asymptote $y = c$ for $x \to +\infty$, then $f(x)$ also has the same asymptote $y = c$ for $x \to -\infty$; however, this is not necessarily true for a general function $f(x)$.

Solution: This is true for rational functions (polynomials over polynomials), but not true in general ($\arctan(x)$).

49. True **FALSE** We often rescale $e^{-\frac{x^2}{2}}$ to $\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ because a function should be more complicated in order to be useful.

Solution: We rescale so the area under the curve is 1.

50. **TRUE** False Similar triangles are a geometric tool that can help us solve some Calculus problems.

Solution: We use them in related rates.

51. True **FALSE** When plugging values for x into the Taylor polynomials of trig. functions, we have to make sure that these values are in degrees, not radians, because it is simpler to work in degrees.

Solution: We need them in radians.

52. True **FALSE** All of the derivatives of the nth Taylor polynomial $T_n(x)$ at $x = a$ of a function $f(x)$ and the function itself are equal at $x = a$.

Solution: Only the 0th through the nth derivatives are equal.

53. True **FALSE** We know that the $\sin(x)$ function is an odd function because it is a strange function and, in addition, all powers of x appearing in its Taylor polynomials at $a = 0$ are of odd degrees.

Solution: We call it odd because $f(-x) = -f(x)$. This means that all powers of x appearing in its Taylor polynomial at $a = 0$ are of odd degree.

54. **TRUE** False Quadratic approximations are, in general, better than linear approximations on larger intervals around the center $x = a$.

Solution: The higher order you get, the better the approximation gets generally.

55. **TRUE** False Calculators use Taylor expansions to compute values of functions that would otherwise be hard or impossible to compute by hand.

56. **TRUE** False Factorials appear in the denominators of terms in Taylor polynomials due to the fact that $(x^n)^{(n)} = n!$ for all $n = 0, 1, 2, \dots$.

57. **TRUE** False L'Hopital's Rule (LH) is a dangerous tool in the hands of people who do not verify its conditions before applying it.

Solution: You need to make sure you get an indeterminate ∞/∞ or $0/0$ before applying it.

58. True **FALSE** If the conditions for L'Hopital's Rule (LH) are satisfied, then we can calculate the given limit by applying LH once or several times.

Solution: Sometimes using LH will not help us solve the problem and we need to simplify like e^x/e^x.

59. True **FALSE** When the first derivative $f'(a) \neq 0$, we can use it to determine if the quadratic approximation of $f(x)$ at $x = a$ tends to be an overestimate or an underestimate of $f(x)$.

Solution: We can use the second derivative to determine whether the linear approximation is an over or underestimate but not the other way around.

60. True **FALSE** We expand $\ln(1 + x)$ at $a = 0$ instead of $\ln x$ at $a = 0$ because we want to practice finding Taylor polynomials on more complicated functions.

Solution: We cannot expand $\ln x$ at $a = 0$ because $\ln 0$ does not exist.

61. True **FALSE** Newton's method is a more sophisticated version of using Taylor Polynomials, since when it does works, it works faster.

Solution: Notions such as sophisticated and faster are not precise or accurate here.

62. **TRUE** False Newton's method utilizes repeatedly the linear approximations (or tangent lines) for $f(x)$, but taken at various points on the graph.

Solution: We find tangents, then find when they intersect the x axis repeatedly.

63. True **FALSE** Newton's method can fail if we chose the initial x_1 too far from the intended root, if in the process some derivative $f'(x_n) = 0$ or does not exist, or for no good reason.

Solution: It occurs in one of those reasons (it cannot fail for no good reason).

64. True **FALSE** The formula for Newton's method involves the same ratio of a function $f(x)$ and its derivative as we saw in the "Black Cloud" example, which also equals the logarithmic derivative of $f(x)$.

Solution: The formula uses $f(x)/f'(x)$, which is the reciprocal of the logarithmic derivative.

65. **TRUE** False $\sqrt{3}$ can be approximated by using Taylor Polynomials and by Newton's method; however, different functions are needed in each approach.

Solution: We use the function $\sqrt{x}$ for Taylor Polynomials and $x^2 - 3$ for Newton's method.

66. True **FALSE** There are no formulas for solving degree 3 polynomial equations, and hence we must use Taylor polynomials to approximate the roots of these polynomials.

Solution: There are formulas but they are too complicated to use.

67. **TRUE** False L'Hospital's Rule (LH) can be used for finding limits in cases of product and exponential indeterminancies, but some preparation work needs to be done to rewrite the problem into a quotient indeterminacy before applying LH.

Solution: We can only use LH when we have ∞/∞ or $0/0$.

68. True **FALSE** If for a function defined and twice differentiable on $\mathbb{R}$ we find out that its first derivative has some root r and its second derivative is everywhere negative, then we can conclude that the function has one local (= global) minimum.

Solution: The function will have one local maximum, not minimum.

69. **TRUE** False Newton's method is useful for approximating critical points of a function.

Solution: We can use Newton's method to find critical points, or when $f'(x) = 0$.

70. **TRUE** False We can rewrite an exponential indeterminacy as a product indeterminacy and then use CR and PR to find the derivative of the original function.